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On the Rate of Convergence of Cesàro Means of Walsh–Fourier Series*

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The aim of this paper is to make a complete investigation concerning the interaction between the rate of convergence of Cesàro means of Walsh–Fourier series and the modulus of continuity. We give the best possible sufficient conditions with respect to the modulus of continuity that implies the convergence at a given rate. We also give the best necessary conditions. These questions are studied in L^p $(1 \leqslant p < \infty)$ and in uniform norms. As a consequence, we receive the best results for the Lipschitz classes. The solution of a problem of F. Móricz and A. H. Siddiqi (1992, *J. Approx. Theory* **70**, 375–389), i.e., the characterization of the Favard (saturation) classes of the Cesàro summation, can be derived from our theorems.

1. INTRODUCTION

Let **N** denote the set of natural numbers, and **P** the set of positive integers. Let r_k represent the kth Rademacher function, i.e.,

$$r_0(x) = \begin{cases} +1 & \text{if } 0 \leqslant x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

periodic with 1, and

$$r_k(x) = r_0(2^k x) \qquad (k \in \mathbf{P}).$$

The Walsh functions in the Paley enumeration can be defined as products of Rademacher functions as

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k},$$

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where $n=\sum_{n=0}^{\infty} n_k 2^k$ ($n_k=0$ or 1, $n\in\mathbf{N}$). The Dirichlet kernels with respect to the Walsh system are defined by the sum

$$D_k=\sum_{j=0}^{k-1} w_j \qquad (k\in\mathbf{P}).$$

It is known that D_{2^n} ($n\in\mathbf{N}$) enjoys the nice property

$$D_{2^n}(x)=\begin{cases}2^n & \text{if } 0\leqslant x<2^{-n},\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

The collection of functions of the form

$$P=\sum_{k=0}^{n-1} a_k w_k$$

with real a_k's, i.e., the set of Walsh polynomials of order less than $n\in\mathbf{P}$ is denoted by $\mathscr{P}_n$. Recall that $\mathscr{P}_{2^n}$ ($n\in\mathbf{N}$) coincides with the set of $\mathscr{A}_n$ measurable real functions, where $\mathscr{A}_n$ denotes the σ algebra generated by the dyadic intervals

$$I_n(k)=[k2^{-n},(k+1)2^{-n}) \qquad (0\leqslant k<2^n, k\in\mathbf{N}).$$

In this paper we study approximation problems in the $L^p=L^p[0,1)$ ($1\leqslant p<\infty$) spaces (with respect to the usual Lebesgue measure). Uniform approximation is studied in C_W. That is the closure of the set of the Walsh polynomials in the uniform norm. In other words C_W consists of the functions continuous at every dyadic irrational of $[0,1)$, continuous from the right at every point of $[0,1)$, which have a finite limit from the left on $(0,1]$.

From now on X^p, $1\leqslant p\leqslant\infty$, denotes L^p if $1\leqslant p<\infty$, and C_W if $p=\infty$. Set

$$\|f\|_p=\left(\int_0^1 |f|^p\right)^{1/p} \qquad (1\leqslant p<\infty),$$

$$\|f\|_\infty=\operatorname{ess\,sup}|f|.$$

The best approximation of an $f\in X^p$ ($1\leqslant p\leqslant\infty$) is defined as

$$E_n(f,X^p)=\inf_{P\in\mathscr{P}_n}\|f-P\|_p \qquad (n\in\mathbf{P}).$$

Any $x\in[0,1)$ can be uniquely written in the form

$$x=\sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where $\liminf_{k\to\infty} x_k = 0$. This is called the dyadic expansion of x. For any $x, y \in [0, 1)$ with dyadic expansions $\sum_{k=0}^{\infty} x_k 2^{-(k+1)}$, $\sum_{k=0}^{\infty} y_k 2^{-(k+1)}$ their dyadic sum is defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Let τ_t $(t \in [0, 1))$ denote the dyadic translation by t, i.e.,

$$\tau_t f(x) = f(x \dotplus t) \qquad (x \in [0, 1), f \in X^1).$$

If $f \in X^p$ $(1 \leqslant p \leqslant \infty)$, then its dyadic X^p modulus of continuity is defined as

$$\omega_p(\delta, f) = \sup_{t<\delta} \|f - \tau_t f\|_p \qquad (\delta > 0).$$

This X^p modulus of continuity has some properties different from the classical one. For instance, $\omega_p(2\delta, f)$ cannot be dominated by $C\omega_p(\delta, f)$ $(C > 0$ absolute constant). However, if $2^{-n-1} \leqslant \delta_1 < \delta_2 < 2^{-n}$ for some $n \in \mathbf{N}$, then

$$\omega_p(\delta_2, f) \leqslant 2\omega_p(\delta_1, f) \qquad (f \in X^p, 1 \leqslant p \leqslant \infty).$$

Consequently, $(\omega_p(2^{-n}, f), n \in \mathbf{N})$ completely characterizes the X^p modulus of continuity of $f \in X^p$.

In particular, (see [3, 8]), for any nonnegative sequence $\omega = (\omega_n, n \in \mathbf{N})$ tending monotonically to 0—in notation $\omega \searrow 0$—there exists $f \in X^p$ such that

$$\omega_p(2^{-n}, f) = \omega_n \qquad (n \in \mathbf{N}).$$

Such an $\omega = (\omega_n, n \in \mathbf{N})$ is called a dyadic modulus of continuity. Let H_p^{ω} denote the Hölder class generated by ω, i.e.,

$$H_p^{\omega} = \{f \in X^p : \omega_p(2^{-n}, f) = O(\omega_n) \text{ as } n \to \infty\}.$$

Similarly, $H_p^{\omega_f}$ $(f \in X^p)$ is the Hölder class generated by $\omega_f = (\omega_p(2^{-n}, f), n \in \mathbf{N})$.

The Walsh–Fourier coefficients of a function $f \in X^1$ are defined by

$$\hat{f}(n) = \int_0^1 f w_n \qquad (n \in \mathbf{N}).$$

The Walsh–Fourier series of f is the series

$$\sum_{n=0}^{\infty} \hat{f}(n) \, w_n.$$

Furthermore, let $S_n f$ denote the nth partial sum of the Walsh–Fourier series of f, i.e.,

$$S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\, w_k \qquad (n \in \mathbf{P}).$$

The following inequalities are due to Watari [12]. These show that there is a strong connection among $\omega_p(2^{-n}, f)$, $E_{2^n}(f, X^p)$, and $\|f - S_{2^n} f\|_p$ $(f \in X^p,\ n \in \mathbf{N})$. In particular,

$$\tfrac{1}{2}\,\omega_p(2^{-n}, f) \leqslant \|f - S_{2^n} f\|_p \leqslant \omega_p(2^{-n}, f), \tag{2}$$

and

$$\tfrac{1}{2}\,\|f - S_{2^n} f\|_p \leqslant E_{2^n}(f, X^p)\|_p \leqslant \|f - S_{2^n} f\|_p. \tag{3}$$

Remark. It is known that the Walsh functions are the characters of the dyadic group G. Consequently, in many problems concerning the Walsh system the structure of the dyadic group plays an important role. Using an almost one-to-one measure preserving map that structure can be transferred to $[0, 1)$. (For details of this correspondence see [9].) This is, for instance, how the concept of the function space C_W arises from the space of continuous functions on G. Although we work on $[0, 1)$ in this paper all the results and methods used can be formulated also on G.

Main Results

The $(C, 1)$-means of an $f \in X^1$ are defined as

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (n \in \mathbf{P}).$$

If K_n denotes the nth Walsh–Fejér kernel, i.e.,

$$K_n = \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (n \in \mathbf{P}),$$

then $\sigma_n f = f * K_n$, where $*$ stands for dyadic convolution.

Yano [13] has proved that $\|K_n\|_1 \leqslant 2$ $(n \in \mathbf{P})$. Consequently, $\|f - \sigma_n f\|_p \to 0$ as $n \to \infty$ $(f \in X^p,\ 1 \leqslant p \leqslant \infty)$. However (see, e.g., [5, 9]), the rate of convergence cannot be better than $O(n^{-1})$ $(n \to \infty)$ for non-constant functions. In the following theorems we use the dyadic X^p modulus of continuity to characterize the set of functions in X^p whose

$(C, 1)$-means converge at a given rate. The rate of convergence is prescribed by a sequence $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$.

As we will see all the necessary, and sufficient conditions with respect to the X^p modulus of continuity in Theorems 2–4 are inequalities, whose left sides are monotonically increasing as $n \to \infty$, while the right side is equal to $(n\alpha_n, n \in \mathbf{N})$. This inspires the idea that we only have to deal with sequences α for which $(k\alpha_k, k \in \mathbf{N})$ is monotonically increasing.

Indeed, the following is true.

THEOREM 1. *Let* $f \in X^p$ $(1 \leqslant p \leqslant \infty)$ *and* $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$. *Then* $\|f - \sigma_n f\|_p = O(\alpha_n)$ *implies* $\|f - \sigma_n f\|_p = O((1/n) \inf_{k > n} k\alpha_k)$ $(k, n \in \mathbf{N}, n \to \infty)$.

Remark. The above theorem includes the solution of the saturation problem with respect to the Cesàro means. Indeed, if $\inf_{k \in \mathbf{N}} k\alpha_k = 0$ then $\|f - \sigma_k f\|_p = O(\alpha_k)$, $k \to \infty$ $(1 \leqslant p \leqslant \infty)$ if and only if f is equivalent to a constant function. In particular [9], if $\|f - \sigma_k f\|_p = o(k^{-1})$, $k \to \infty$, then f is necessarily constant.

We have formalized our main results in three theorems.

THEOREM 2. *Let* $2 \leqslant p < \infty$ *and* $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$.

(i) *If for an* $f \in X^p$

$$\left(\sum_{k=1}^{n} k\omega_p^2(1/k, f) \right)^{1/2} = O(n\alpha_n) \qquad (n \to \infty)$$

then

$$\|f - \sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty).$$

(ii) *If* ω *is a dyadic modulus of continuity for which*

$$\left(\sum_{k=1}^{n} k\omega_k^2 \right)^{1/2} \neq O(n\alpha_n) \qquad (n \to \infty)$$

then there exists $f \in H_p^\omega$ *such that*

$$\|f - \sigma_n f\|_p \neq O(\alpha_n) \qquad (n \to \infty).$$

(iii) *If for an* $f \in X^p$

$$\|f - \sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty)$$

then

$$\left(\sum_{k=1}^{n} k^{p-1}\omega_p^p(1/k, f)\right)^{1/2} = O(n\alpha_n) \qquad (n \to \infty).$$

(iv) *Let*

$$\Omega_p = \left\{ f \in X^p : \left(\sum_{k=1}^{n} k^{p-1}\omega_p^p(1/k, f)\right)^{1/p} = O(n\alpha_n)\ (n \to \infty) \right\}$$

and

$$\Sigma_p = \{ f \in X^p : \|f - \sigma_n f\|_p = O(\alpha_n)\ (n \to \infty) \Big\}.$$

Then

$$\bigcup_{f \in \Sigma_p} H_p^{\omega_f} = \Omega_p.$$

In the following theorem $1 < p \leqslant 2$, and the corresponding results are dual to those in Theorem 2.

THEOREM 3. *Let* $1 < p \leqslant 2$ *and* $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$.

(i) *If for an* $f \in X^p$

$$\left(\sum_{k=1}^{n} k^{p-1}\omega_p^p(1/k, f)\right)^{1/p} = O(n\alpha_n) \qquad (n \to \infty)$$

then

$$\|f - \sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty).$$

(ii) *If* ω *is a dyadic modulus of continuity for which*

$$\left(\sum_{k=1}^{n} k^{p-1}\omega_k^p\right)^{1/p} \neq O(n\alpha_n) \qquad (n \to \infty)$$

then there exists $f \in H_p^{\omega}$ *such that*

$$\|f - \sigma_n f\|_p \neq O(\alpha_n) \qquad (n \to \infty).$$

(iii) *If for an* $f \in X^p$

$$\|f - \sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty)$$

then

$$\left(\sum_{k=1}^{n} k\omega_p^2(1/k, f)\right)^{1/2} = O(n\alpha_n) \qquad (n \to \infty).$$

(iv) *Let*

$$\Omega_p^* = \left\{ f \in X^p : \left(\sum_{k=1}^{n} k\omega_p^2(1/k, f) \right)^{1/2} = O(n\alpha_n)\ (n \to \infty) \right\}$$

and

$$\Sigma_p = \{ f \in X^p : \|f - \sigma_n f\|_p = O(\alpha_n)\ (n \to \infty) \}.$$

Then

$$\bigcup_{f \in \Sigma_p} H_p^{\omega_f} = \Omega_p^*.$$

The next cases, i.e., when $p = 1$ or ∞, are slightly different from the previous cases.

THEOREM 4. *Let $p = 1$ or ∞, and $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$.*

(i) *If for an $f \in X^p$*

$$\sum_{k=1}^{n} \omega_p(1/k, f) = O(n\alpha_n)$$

then

$$\|f - \sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty).$$

(ii) *If ω is a dyadic modulus of continuity for which*

$$\sum_{k=1}^{n} \omega_k \neq O(n\alpha_n)$$

then there exists $f \in H_p^\omega$ such that

$$\|f - \sigma_n f\|_p \neq O(\alpha_n) \qquad (n \to \infty).$$

(iii) *If for an $f \in X^p$*

$$\|f - \sigma_n f\|_p = O(\alpha_n)$$

then

$$\max_{0 < k < n} k\omega_p(1/k, f) = O(n\alpha_n) \qquad (n \to \infty).$$

(iv) *Let*

$$\Omega_p^{**} = \{ f \in X^p : \max_{0 < k < n} k\omega_p(1/k, f) = O(n\alpha_n)\ (n \to \infty) \}$$

and

$$\Sigma_p = \{f \in X^p : \|f - \sigma_n f\|_p = O(\alpha_n)\ (n \to \infty)\}.$$

Then

$$\bigcup_{f \in \Sigma_p} H_p^{\omega_f} = \Omega_p^{**}.$$

Remarks. 1. If a growth order condition with respect to the X^p modulus of continuity is satisfied by an $f \in X^p$ then the same holds for any element of the Hölder class generated by f. This means that the best possible sufficient condition is the one which induces the largest subset of the set of Hölder classes contained by—using the notation of the above theorems—Σ_p. Similarly, the best necessary condition is the one that induces the smallest subset of the set of Hölder classes which covers Σ_p. The (ii) (resp. (iv)) parts of Theorems 2–4 show that the corresponding sufficient (resp. necessary) conditions in (i) (resp. (iii)) are the best possible in this terminology.

2. Theorem 1 together with Theorems 2–4 has the following consequence. If α, $\beta \searrow 0$ with $\inf_{k>n} k\alpha_k = O(n\beta_n)$, $n \to \infty$, then $\|f - \sigma_n f\|_p = O(\alpha_n)$ implies $\|f - \sigma_n f\|_p = O(\beta_n)$ $(1 \leqslant p \leqslant \infty)$. Moreover, if at the same time $\inf_{k>n} k\beta_k \neq O(n\alpha_n)$, $n \to \infty$, then there exists $f \in X^p$ with $\|f - \sigma_n f\|_p = O(\beta_n)$ and $\|f - \sigma_n f\|_p \neq O(\alpha_n)$.

AUXILIARIES

In order to prove our theorems we need some preliminary results and lemmas. C will denote an absolute positive constant and A_p, B_p, depending only on p, denote positive constants, not necessarily the same in different occurrences.

Let $\mathbf{L}^0$ denote the collection of sequences $\mathbf{g} = (g_k, k \in \mathbf{N})$, where each g_k is a real valued measurable function defined on $[0, 1)$. For any $1 \leqslant p$, $q \leqslant \infty$ denote by $L^p(l^q)$, resp. $l^p(L^q)$, the Banach spaces of $\mathbf{g} \in \mathbf{L}^0$ for which

$$\|\mathbf{g}\|_{L^p(l^q)} = \left\| \left(\sum_{k=0}^{\infty} |g_k|^q \right)^{1/q} \right\|_p,$$

resp.

$$\|\mathbf{g}\|_{l^p(L^q)} = \left(\sum_{k=0}^{\infty} \|g_k\|_q^p \right)^{1/p},$$

is finite, and these quantities serve as norms in them. (If $p, q=\infty$ then we make the obvious modifications.) Concerning the properties of these spaces we refer to [10]. Set

$$\Delta_k f = S_{2^{k+1}} f - S_{2^k} f \qquad (f \in L^1),$$

and

$$\boldsymbol{\Delta} f = (\Delta_k f, k \in \mathbf{N}).$$

Clearly $\boldsymbol{\Delta} f \in \mathbf{L}^0$. The quadratic variation of an $f \in L^1$ is defined as

$$Qf = \left(\sum_{k=0}^{\infty} |\Delta_k f|^2 \right)^{1/2}.$$

Using the above notations we have $\|Qf\|_p = \|\boldsymbol{\Delta} f\|_{L^p(l^2)}$. It is known [7] by Paley's inequality that the norms $\|Qf\|_p$ and $\|f\|_p$ are equivalent for any $1<p<\infty$, $f \in L^p$, i.e.,

$$A_p \|\boldsymbol{\Delta} f\|_{L^p(l^2)} \leqslant \|f\|_p \leqslant B_p \|\boldsymbol{\Delta} f\|_{L^p(l^2)}. \tag{4}$$

The following inequalities, which are applied frequently in the sequel, are immediate consequences of (1).

$$\|\Delta_k f\|_p \leqslant \|f\|_p \quad \text{and} \quad \|S_{2^k} f\|_p \leqslant \|f\|_p \quad (k \in \mathbf{N}, f \in X^p, 1 \leqslant p \leqslant \infty). \tag{5}$$

Suppose $\mathbf{g} \in \mathbf{L}^0$ and $f \in L^p$ $(2 \leqslant p \leqslant \infty)$. Then

$$\|\mathbf{g}\|_{L^p(l^2)} \leqslant \|\mathbf{g}\|_{l^2(L^p)}, \tag{6}$$

and

$$\|\boldsymbol{\Delta} f\|_{L^p(l^p)} \leqslant \|f\|_p. \tag{7}$$

The case $p=\infty$ is trivial in (6). If $p<\infty$ then use the Minkowski inequality for the $L^{p/2}$ space to see that

$$\|\mathbf{g}\|_{L^p(l^2)} = \left\| \sum_{k=0}^{\infty} \mathbf{g}_k^2 \right\|_{p/2}^{1/2} \leqslant \left(\sum_{k=0}^{\infty} \|\mathbf{g}_k^2\|_{p/2} \right)^{1/2} = \|\mathbf{g}\|_{l^2(L^p)}.$$

We show (7) by using an interpolation with respect to the $L^p(l^q)$ spaces. We note that $f \to \boldsymbol{\Delta} f$ is a linear map from L^p to $\mathbf{L}^0$. Therefore, by the interpolation theorem of [9] it is enough to show that $\boldsymbol{\Delta}$ is of type $(L^2, L^2(l^2))$ and $(L^\infty, L^\infty(l^\infty))$. Obviously, $\|\boldsymbol{\Delta} f\|_{L^2(l^2)} = \|f\|_{L^2}$. Since $\|\Delta_k f\|_\infty \leqslant \|f\|_\infty$ $(k \in \mathbf{N})$, we have by (5)

$$\|\boldsymbol{\Delta} f\|_{L^\infty(l^\infty)} = \|\boldsymbol{\Delta} f\|_{l^\infty(L^\infty)} \leqslant \|f\|_{L^\infty}.$$

Now let $1 \leqslant p \leqslant 2$. Then

$$\|\mathbf{g}\|_{L^p(l^2)} \geqslant \|\mathbf{g}\|_{l^2(L^p)}, \tag{8}$$

and

$$\|\Delta f\|_{L^p(l^p)} \geqslant \|f\|_{L^p}. \tag{9}$$

It is easy to see that similarly to the L^p cases the space dual to $L^p(L^q)$ is $L^{p'}(l^{q'})$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$ $(1 \leqslant p,\ q < \infty)$. Moreover, the Riesz-representation theorem for this case is of the form

$$\|\mathbf{f}\|_{L^p(l^q)} = \sup\left\{\int_0^1 \sum_{k=0}^{\infty} f_k g_k : \|\mathbf{g}\|_{L^{p'}(l^{q'})} \leqslant 1\right\}.$$

Similar relations hold for the $l^p(L^q)$ spaces. Therefore, (8) follows from (6) by duality arguments. Furthermore, (7) implies

$$\begin{aligned}\|f\|_p &= \sup\left\{\int_0^1 \sum_{k=0}^{\infty} \Delta_k f \Delta_k g : \|g\|_{p'} \leqslant 1\right\}\\ &\leqslant \sup\left\{\int_0^1 \sum_{k=0}^{\infty} \Delta_k f \Delta_k g : \|\Delta g\|_{L^{p'}(l^{p'})} \leqslant 1\right\}\\ &\leqslant \|\Delta f\|_{L^p(l^p)} \qquad (1/p + 1/p' = 1,\ 1 \leqslant p \leqslant 2).\end{aligned}$$

Let d denote the restriction of the operator of the dyadic derivative onto $\mathscr{P}$, i.e., if $P_n = \sum_{k=0}^{n-1} a_k w_k \in \mathscr{P}_n$ $(n \in \mathbf{P})$ then

$$dP_n = \sum_{k=0}^{n-1} k a_k w_k.$$

We refer to [9] for details. It is known [1] that there are inequalities with respect to the dyadic derivative which are similar to the classical Bernstein and Jackson inequalities. They are of the form on $\mathscr{P}$

$$\begin{aligned}&\|dP_n\|_p \leqslant 2n\ \|P_n\|_p,\\ &E_k(P_n, X^p) \leqslant A_p k^{-1}\ \|dP_n\|_p \qquad (P_n \in \mathscr{P}_n,\ n, k \in \mathbf{P}).\end{aligned}$$

In particular, we have by (2) and (3) that

$$\tfrac{1}{4} 2^{-k}\ \|d\Delta_k f\|_p \leqslant \|\Delta_k f\|_p \leqslant A_p 2^{-k}\ \|d\Delta_k f\|_p \qquad (f \in X^p,\ 1 \leqslant p \leqslant \infty,\ k \in \mathbf{N}). \tag{10}$$

The following lemma shows that the investigation of convergence of Cesàro means can be restricted to indices of the form 2^n ($n \in \mathbf{N}$).

LEMMA 1. *Let* $f \in X^p$ $(1 \leqslant p \leqslant \infty)$. *Then*

$$\|f - \sigma_{2^{n+1}} f\|_p \geqslant A_p \, \|f - \sigma_{2^n} f\|_p,$$

and

$$\|f - \sigma_{2^n + k} f\|_p \leqslant A_p \, \|f - \sigma_{2^n} f\|_p \qquad (k, n \in \mathbf{N}, 0 \leqslant k \leqslant 2^n).$$

Proof of Lemma 1. We have by definition and by (5) that

$$\begin{aligned} \|f - \sigma_{2^{n+1}} f\|_p &\geqslant \|S_{2^n}(f - \sigma_{2^{n+1}} f)\|_p = \tfrac{1}{2} \, \|S_{2^n}(f - \sigma_{2^n} f)\|_p \\ &\geqslant \tfrac{1}{2} \, (\|f - \sigma_{2^n} f\|_p - \|\varDelta_n f\|_p \\ &\quad - \|f - S_{2^{n+1}} f\|_p) \qquad (f \in X^p, n \in \mathbf{N}). \end{aligned}$$

Since $\varDelta_n(f - \sigma_{2^{n+1}} f) = 2^{-n-1} d\varDelta_n f$ it follows from (10) and (5) that

$$\|\varDelta_n f\|_p \leqslant A_p \, \|f - \sigma_{2^{n+1}} f\|_p.$$

Similarly,

$$\begin{aligned} \|f - S_{2^{n+1}} f\|_p &= \|(f - \sigma_{2^{n+1}} f) - S_{2^{n+1}}(f - \sigma_{2^{n+1}} f)\|_p \\ &\leqslant 2 \, \|f - \sigma_{2^{n+1}} f\|_p. \end{aligned}$$

Combining these inequalities we have

$$\|f - \sigma_{2^{n+1}} f\|_p \geqslant A_p \, \|f - \sigma_{2^n} f\|_p.$$

On the other hand [2] for all $k, n \in \mathbf{N}$, $0 \leqslant k \leqslant 2^n$,

$$(2^n + k) \, K_{(2^n + k)} = 2^n K_{2^n} + k D_{2^n} + w_{2^n} k K_k.$$

This implies

$$\|f - \sigma_{2^n + k} f\|_p \leqslant \|f - \sigma_{2^n} f\|_p + \|f - S_{2^n} f\|_p + \|\varDelta_n f * w_{2^n} K_k\|_p.$$

Since

$$\|\varDelta_n f * w_{2^n} K_k\|_p \leqslant \|w_{2^n} K_k\|_1 \, \|\varDelta_n f\|_p \leqslant 2 \, \|f - S_{2^n} f\|_p,$$

we have by the same estimations as those above that

$$\|f - \sigma_{2^n + k} f\|_p \leqslant A_p \, \|f - \sigma_{2^n} f\|_p \qquad (k, n \in \mathbf{N}, 0 \leqslant k \leqslant 2^n). \quad \blacksquare$$

In the following two corollaries let $f \in X^p$ $(1 \leqslant p \leqslant \infty)$ and $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$ be given.

COROLLARY 1. *$\|f - \sigma_k f\|_p = O(\alpha_k)$ is equivalent to $\|f - \sigma_{2^k} f\|_p = O(\alpha_{2^k})$ $(k \to \infty)$.*

This is an immediate consequence of Lemma 1.

COROLLARY 2.

$$\|f - \sigma_k f\|_p = O(\alpha_k) \qquad (k \to \infty) \tag{11}$$

if and only if

$$\|dS_{2^n} f\|_p = O(2^n \alpha_{2^n}) \qquad \textit{and} \qquad E_{2^n}(f, X^p) = O(\alpha_{2^n}) \quad (n \to \infty) \tag{12}$$

hold.

Indeed, it is easy to see by definition that

$$2^n K_{2^n} = 2^n D_{2^n} - dD_{2^n} \qquad (n \in \mathbf{N}).$$

Hence by (2), (3)

$$\|dS_{2^n} f\|_p \leqslant 2^n \|f - \sigma_{2^n} f\|_p + 2^n \|f - S_{2^n} f\|_p \leqslant 3 \cdot 2^n \|f - \sigma_{2^n} f\|_p,$$

and

$$2^n \|f - \sigma_{2^n} f\|_p \leqslant \|dS_{2^n} f\|_p + 2 \cdot 2^n E_{2^n}(f, X^p).$$

Consequently, by Corollary 1, (11) follows from (12). To complete the proof observe that (11) implies $E_{2^n}(f, X^p) = O(\alpha_{2^n})$ $(n \to \infty)$. ∎

Corollary 2 shows the strong relation between the Cesàro means and the dyadic derivative. For the general theory with respect to such relations we refer to [11].

The following inequality will be used frequently. If $f \in X^p$ $(1 \leqslant p \leqslant \infty)$ and $1 \leqslant q < \infty$ then for any $n \in \mathbf{N}$

$$\left(\sum_{k=0}^{n} (2^k \|\Delta_k f\|_p)^q \right)^{1/q} \geqslant 1/4 \left(\sum_{k=0}^{n} (2^k \omega_p(2^{-k}, f))^q \right)^{1/q} - 2^n \omega_p(2^{-n-1}, f). \tag{13}$$

Indeed, since $\|\Delta_k f\|_p \geqslant \|f - S_{2^k} f\|_p - \|f - S_{2^{k+1}} f\|_p$ it is clear by (2) and the triangle inequality that

$$\left(\sum_{k=0}^{n} (2^k \|\Delta_k f\|_p)^q\right)^{1/q} \geqslant \left(\sum_{k=0}^{n} (2^k(\|f-S_{2^k}f\|_p - \|f-S_{2^{k+1}}f\|_p))^q\right)^{1/q}$$

$$\geqslant \left(\sum_{k=0}^{n} (2^k \|f-S_{2^k}f\|_p)^q\right)^{1/q}$$

$$-\left(\sum_{k=1}^{n} (2^{k-1} \|f-S_{2^k}f\|_p)^q\right)^{1/q} - 2^n \|f-S_{2^{n+1}}f\|_p$$

$$\geqslant 1/2\left(\sum_{k=0}^{n} (2^k \|f-S_{2^k}f\|_p)^q\right)^{1/q} - 2^n \|f-S_{2^{n+1}}f\|_p$$

$$\geqslant 1/4\left(\sum_{k=0}^{n} (2^k\omega_p(2^{-k},f))^q\right)^{1/q} - 2^n\omega_p(2^{-n-1},f).$$

LEMMA 2. *Let* $f=\sum_{n=0}^{\infty} a_n r_n \in X^p$ $(1 \leqslant p < \infty)$ *and* $\alpha = (\alpha_k, k \in \mathbf{N}) \searrow 0$. *Then*

$$\|f-\sigma_n f\|_p = O(\alpha_n) \qquad (n \to \infty)$$

if and only if

$$\left(\sum_{k=1}^{n} k\omega_p^2(1/k, f)\right)^{1/2} = O(n\alpha_n) \quad (n \to \infty).$$

Proof of Lemma 2. First let, $f \in X^p$ such that

$$\left(\sum_{k=1}^{n} (2^k\omega_p(2^{-k},f))^2\right)^{1/2} = O(2^n\alpha_{2^n}) \qquad (n \to \infty).$$

Clearly, $E_{2^n}(f, X^p) \leqslant \omega_p(2^{-n}, f) = O(\alpha_{2^n})$ $(n \to \infty)$. Note by definition that $dS_{2^n}f = \sum_{k=1}^{n-1} 2^k a_k r_k$ $(n \in \mathbf{P})$. Applying Kintchin's (see, e.g., [9]) inequality we have

$$\|dS_{2^n}f\|_p \leqslant A_p\left(\sum_{k=1}^{n-1} (2^k a_k)^2\right)^{1/2} = A_p\left(\sum_{k=1}^{n-1} (2^k \|\Delta_k f\|_p)^2\right)^{1/2}$$

$$\leqslant A_p\left(\sum_{k=1}^{n} (2^k\omega_p(2^{-k},f))^2\right)^{1/2} \leqslant A_p 2^n\alpha_{2^n} \qquad (n \in \mathbf{P}).$$

Hence, $\|f-\sigma_n f\|_p = O(\alpha_n)$ $(n \to \infty)$ follows from Corollary 2.

Now let $f \in X^p$ such that $\|f-\sigma_n f\|_p = O(\alpha_n)$ $(n \to \infty)$. Applying (2), (13), and, again, Kintchin's inequality we obtain

$$\left(\sum_{k=1}^{n}(2^k\omega_p(2^{-k},f))^2\right)^{1/2}\leqslant 4\left(\sum_{k=1}^{n}(2^k\,\|\Delta_k f\|)^2\right)^{1/2}+4\cdot 2^n\omega_p(2^{-n-1},f)$$

$$=4\left(\sum_{k=1}^{n-1}(2^k a_k)^2\right)^{1/2}+4\cdot 2^n\omega_p(2^{-n-1},f)$$

$$\leqslant A_p(\|dS_{2^n}f\|_p+2^nE_{2^n}(f,X^p))\qquad (n\in\mathbf{P}).$$

The proof can be completed by Corollary 2. ∎

Proofs

Proof of Theorem 1. Applying Corollary 1 we need only prove that $\|f-\sigma_{2^n}f\|_p=O(\alpha_{2^n})$ implies $\|f-\sigma_{2^n}f\|_p=O((1/2^n)\inf_{k>n}2^k\alpha_{2^k})$. Since $\|dS_{2^n}f\|_p=O(2^n\alpha_{2^n})$ increases as $n\to\infty$ it can only be dominated by $C2^n\alpha_{2^n}$ if

$$\|dS_{2^n}f\|_p=O(\inf_{k\geqslant n}2^k\alpha_{2^k})\qquad (n\to\infty).\tag{14}$$

Let $\|f-S_{2^n}f\|_p=\beta_{2^n}=O(\alpha_{2^n})$. If $2^n\beta_{2^n}$ is quasi-monotonically increasing then the proof can be completed by Corollary 2. If this is not the case, then let $\ell_0=1$ and

$$\ell_k=\min\{j>\ell_{k-1}:2^j\beta_{2^j}<2^{j-1}\beta_{2^{j-1}}\}\qquad (k\in\mathbf{P}).$$

Observe that

$$\|dS_{2^{\ell_k}}f\|_p\geqslant\|d\Delta_{\ell_k-1}f\|_p\geqslant C_p2^{\ell_k-1}\,\|\Delta_{\ell_k-1}f\|_p$$

$$\geqslant 2^{\ell_k-1}(\beta_{2^{\ell_k-1}}-\beta_{2^{\ell_k}})\geqslant\tfrac{1}{2}\,2^{\ell_k-1}\beta_{2^{\ell_k-1}}\qquad (k\in\mathbf{N}).$$

Therefore, we have by (14) that

$$2^{\ell_k-1}\beta_{2^{\ell_k-1}}\leqslant A_p\inf_{n\geqslant\ell_k}2^n\alpha_{2^n}\qquad (k\in\mathbf{N}).$$

On the other hand, for any $j\in\mathbf{N}$ there exists

$$N_j=\min\{k\in\mathbf{N}:\ell_k>j\}.$$

Then

$$2^j\beta_{2^j}\leqslant 2^m\beta_m\leqslant C2^m\alpha_m\qquad (j\leqslant m<\ell_{N_j}),$$

and

$$2^j\beta_{2^j}\leqslant 2^{\ell_{N_j}-1}\beta_{2^{\ell_{N_j}-1}}\leqslant C_p\inf_{n\geqslant\ell_{N_j}}2^n\alpha_{2^n}(j\in\mathbf{N}).$$

Consequently,

$$\beta_{2^j} \leqslant A_p \frac{1}{2_j} \inf_{n \geqslant j} 2^n \alpha_{2^n} \qquad (j \in \mathbf{N}).$$

We can finish the proof by Corollary 2. ∎

Proof of Theorem 2. Let $f \in X^p$ such that

$$\left(\sum_{k=1}^{n} (2^k \omega_p(2^{-k}, f))^2 \right)^{1/2} = O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

Consequently (see (2) and (3)), $E_{2^n}(f, X^p) = O(\alpha_{2^n})$ $(n \to \infty)$. We have by (2) and (10) that

$$\|d(\Delta_k f)\|_p \leqslant 4 \cdot 2^k \, \|\Delta_k f\|_p \leqslant 4 \cdot 2^k \omega_p(2^{-k}, f).$$

Therefore, it follows from (6), (4), and (2) that

$$\begin{aligned} \|dS_{2^n} f\|_p &\leqslant A_p \, \|\boldsymbol{\Delta}(dS_{2^n} f)\|_{X^p(l^2)} \\ &\leqslant A_p \, \|\boldsymbol{\Delta}(dS_{2^n} f)\|_{l^2(X^p)} = A_p \left(\sum_{k=1}^{n-1} \|d(\Delta_k f)\|_p^2 \right)^{1/2} \\ &\leqslant 4A_p \left(\sum_{k=1}^{n-1} (2^k \omega_p(2^{-k}, f))^2 \right)^{1/2} = O(2^n \alpha_{2^n}) \qquad (n \in \mathbf{P}). \end{aligned}$$

In view of Corollary 2, this completes the proof of part (i).

To prove (ii) let $\omega = (\omega_k, k \in \mathbf{N}) \searrow 0$ for which

$$\left(\sum_{k=1}^{\infty} (2^k \omega_k)^2 \right)^{1/2} \neq O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

Define $f \in X^p$ by the series

$$f = \sum_{k=0}^{\infty} a_k r_k,$$

where $a_k = (\omega_k^2 - \omega_{k+1}^2)^{1/2}$ $(k \in \mathbf{N})$. Thus

$$\left(\sum_{j=k}^{\infty} a_k^2 \right)^{1/2} = \omega_k \qquad (k \in \mathbf{N}).$$

Therefore, Kintchin's inequality and (2) imply that $f \in X^p$ and

$$A_p \omega_k \leqslant \omega_p(2^{-k}, f) \leqslant B_p \omega_k \qquad (k \in \mathbf{N}).$$

Consequently, $f \in H_p^{\omega}$, and

$$\left(\sum_{k=1}^{n} (2^k \omega_p(2^{-k}, f))^2\right)^{1/2} \neq O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

The proof of (ii) can be completed by Lemma 2.

The hypothesis of (iii) implies $\|dS_{2^n} f\|_p = O(2^n \alpha_{2^n})$ and $E_{2^n}(f_1 X^n) = O(\alpha_{2^n})$ $(n \to \infty)$. Applying (10) and (13) we obtain

$$\begin{aligned}\left(\sum_{k=0}^{n} (2^k \omega_p(2^{-k}, f))^p\right)^{1/p} &\leqslant 4\left(\sum_{k=0}^{n} (2^k \|\Delta_k f\|_p))^p\right)^{1/p} + 4 \cdot 2^n \omega_p(2^{-n-1}, f) \\ &\leqslant A_p \left(\sum_{k=0}^{n} \|d\Delta_k f\|_p^p\right)^{1/p} + 4 \cdot 2^n \omega_p(2^{-n-1}, f) \\ &= A_p \|\Delta(dS_{2^n} f)\|_{X^{p(l^p)}} + 4 \cdot 2^n \omega_p(2^{-n-1}, f).\end{aligned}$$

Therefore, it follows from (7) that

$$\left(\sum_{k=0}^{n} (2^k \omega_p(2^{-k}, f))^p\right)^{1/p} = O(2^n \alpha_{2^n}) \qquad (n \in \mathbf{N}),$$

which is equivalent to the statement of (iii).

Finally, to the proof of (iv) we must show that for any $g \in \Omega_p$ there exists $f \in \Sigma_p$ such that $g \in H_p^{\omega_f}$. To this end let g be an arbitrary element of Ω_p and introduce the notation $\omega_j = \omega_p(2^{-j}, g)$ $(j \in \mathbf{N})$.

Define $(\ell_k, k \in \mathbf{N})$ as

$$\ell_0 = 0, \qquad \ell_{k+1} = \min\{j \in \mathbf{N} : \omega_j \leqslant \tfrac{1}{2}\omega_{\ell_k}\} \quad (k \in \mathbf{N}). \tag{15}$$

Set

$$f_k = a_k \mathscr{I}(\tau_{2^{-(\ell_{k+1}-1)}}(D_{2^{\ell_{k+1}}} - D_{2^{\ell_{k+1}-1}})), \tag{16}$$

where a_k is determined by the identity $\|f_k\|_p = \omega_{\ell_k}$ $(k \in \mathbf{N})$. Here $\mathscr{I}$ denotes the inverse of the dyadic operator d. Let $f = \sum_{k=0}^{\infty} f_k$. Then, we have by the definition of $(\ell_k, k \in \mathbf{N})$ and (2) that for $\ell_k \leqslant n < \ell_{k+1}$

$$\omega_p(2^{-n}, f) \leqslant 2\sum_{j=k}^{\infty} \|f_j\|_p = 2\sum_{j=k}^{\infty} \omega_{\ell_j} \leqslant 4\omega_{\ell_k} \leqslant 8\omega_n,$$

i.e., $f \in \Omega_p$. On the other hand by (2) and (5)

$$\begin{aligned}\omega_p(2^{-n}, f) &\geqslant \omega_p(2^{-(\ell_{k+1}-1)}, f) \geqslant \|f - S_{2^{\ell_{k+1}-1}} f\|_p \\ &\geqslant \|\Delta_{\ell_{k+1}-1} f\|_p = \|f_k\|_p = \omega_{\ell_k} \geqslant \omega_n.\end{aligned}$$

Consequently, $\omega_p(2^{-n}, f) \geqslant \omega_p(2^{-n}, g)$. In particular, $g \in H_p^{\omega_f}$.

It remains to establish $f \in \Sigma_p$. Let $\ell_k \leqslant n < \ell_{k+1}$ $(n, k \in \mathbf{N})$ and note by definition that the supports of df_j's are disjoint, i.e., $\operatorname{supp} df_j \cap \operatorname{supp} df_i = \varnothing$ $(i, j \in \mathbf{N},\ i \neq j)$. Then we have by (10) that

$$\|dS_{2^n} f\|_p^p = \sum_{j=0}^{k-1} \|df_j\|_p^p \leqslant 4^p \sum_{j=0}^{k-1} (2^{\ell_j+1} \|f_j\|_p)^p = 4^p \sum_{j=0}^{k-1} (2^{\ell_j+1} \omega_{\ell_j})^p.$$

It follows from the construction of $(\ell_k, k \in \mathbf{N})$ that

$$(2^{\ell_j+1} \omega_{\ell_j})^p \leqslant 4^p \sum_{m=\ell_j}^{\ell_{j+1}-1} (2^m \omega_m)^p \qquad (j \in \mathbf{N}).$$

Consequently,

$$\|dS_{2^n} f\|_p \leqslant 16 \left(\sum_{m=1}^{n} (2^m \omega_m)^p \right)^{1/p} = O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

On the other hand, $E_{2^n}(f, X^p) = O(\alpha_{2^n})$ follows easily from the fact that $f \in \Omega_p$. This completes the proof of (iv). ∎

Proof of Theorem 3. Let $f \in X^p$. Thus by (2), (9), and (10)

$$\begin{aligned} \|dS_{2^n} f\|_p &\leqslant \|\Delta\, dS_{2^n} f\|_{X^p(l^p)} = \|\Delta\, dS_{2^n} f\|_{l^p(X^p)} \\ &\leqslant 4 \left(\sum_{k=0}^{n-1} (2^k \|\Delta_k f\|_p)^p \right)^{1/p} \\ &\leqslant 4 \left(\sum_{k=0}^{n} (2^k \omega_p(2^{-k}, f))^p \right)^{1/p} \qquad (n \in \mathbf{P}). \end{aligned}$$

In particular, if f satisfies the condition of (i) then

$$\|dS_{2^n} f\|_p = O(2^n \alpha_{2^n}) \qquad \text{and} \qquad E_{2^n}(f, X^p) = O(\alpha_{2^n}) \qquad (n \to \infty).$$

Hence, by Corollary 2 we have $\|f - \sigma_k f\|_p = O(\alpha_k)$ $(k \to \infty)$. The statement of (i) is proved.

Next, let $\omega = (\omega_k, k \in \mathbf{N}) \searrow 0$ such that $(\sum_{k=0}^{n} (2^k \omega_k)^p)^{1/p} \neq O(2^n \alpha_{2^n})$ $(n \to \infty)$. Define ℓ_k, f_k $(k \in \mathbf{N})$ and $f \in X^p$ as in (15) and (16). Set $\ell_k \leqslant n < \ell_{k+1}$ $(n, k \in \mathbf{N})$. Then similarly to the proof of (iv) of Theorem 2 we obtain $\omega_p(2^{-n}, f) \leqslant 8\omega_n$. Hence $f \in H_p^\omega$. Introducing the sequence $\beta_n = (1/n) \inf_{k > n} k\alpha_k$ $(n \in \mathbf{N})$ we obviously have $(\sum_{k=0}^{n} (2^k \omega_k)^p)^{1/p} \neq O(2^n \beta_{2^n})$ $(n \to \infty)$. By the definition of ℓ_j it is easy to see that

$$\left(\sum_{j=0}^{n} (2^j \omega_j)^p \right)^{1/p} \leqslant 2 \left(\sum_{j=0}^{k-1} (2^{\ell_j+1} \omega_{\ell_j})^p \right)^{1/p} + 2 \cdot 2^n \omega_{\ell_k} \qquad (\ell_k \leqslant n < \ell_{k+1}, k \in \mathbf{N}).$$

Since $2^j\beta_{2^j}$ increases, and $\beta_{2^j} \searrow 0$ as $j \to \infty$ it follows from the above estimation that

$$\frac{1}{2^n\beta_{2^n}}\left(\sum_{j=0}^{n}(2^j\omega_j)^p\right)^{1/p} \leqslant 2\,\frac{1}{2^{\ell_k}\beta_{2^{\ell_k}}}\left(\sum_{j=0}^{k-1}(2^{\ell_j+1}\omega_{\ell_j})^p\right)^{1/p} + 2\,\frac{1}{2^{\ell_{k+1}}\beta_{2^{\ell_{k+1}}}}\left(\sum_{j=0}^{k}(2^{\ell_j+1}\omega_{\ell_j})^p\right)^{1/p}.$$

Consequently,

$$\limsup_{k\to\infty}\frac{1}{2^{\ell_k}\beta_{2^{\ell_k}}}\left(\sum_{j=0}^{k-1}(2^{\ell_j+1}\omega_{\ell_j})^p\right)^{1/p} = \infty. \tag{17}$$

On the other hand, it follows from the definition of f_j, ℓ_j and from (10) that

$$\|dS_{2^{\ell_k}}f\|_p^p = \sum_{j=0}^{k-1}\|df_j\|_p^p \geqslant A_p\sum_{j=0}^{k-1}(2^{\ell_j+1}\|f_j\|_p)^p = A_p\sum_{j=0}^{k-1}(2^{\ell_j+1}\omega_{\ell_j})^p.$$

Hence we conclude from (17) that $\|dS_{2^n}f\|_p \neq 0(2^n\beta_{2^n})$ $(n\to\infty)$. The proof of (ii) can be completed by Theorem 1.

To verify (iii) suppose that $f \in X^p$ such that $\|f-\sigma_k f\|_p = O(\alpha_k)$ $(k\to\infty)$. Then we have by (10), (9), (8), and (4)

$$\left(\sum_{k=0}^{n-1}(2^k\|\Delta_k f\|_p)^2\right)^{1/2} \leqslant A_p\left(\sum_{k=0}^{n-1}\|\Delta_k(dS_{2^n}f)\|_p^2\right)^{1/2} = A_p\,\|\boldsymbol{\Delta}(dS_{2^n}f)\|_{l^2(X^p)}$$
$$\leqslant A_p\,\|\boldsymbol{\Delta}(dS_{2^n}f)\|_{X^p(l^2)} \leqslant A_p\,\|dS_{2^n}f\|_p \qquad (n\in\mathbf{P}).$$

The inequality (13) and Corollary 2 imply the statement of (iii).

It remains to establish (iv). To this end let $g\in\Omega_p^*$, and use the notation $\omega_n = \omega_p(2^{-n}, g)$ $(n\in\mathbf{N})$. Set

$$f = \sum_{k=0}^{\infty} a_k r_k,$$

where $a_k = (\omega_k^2 - \omega_{k+1}^2)^{1/2}$ $(k\in\mathbf{N})$. Thus

$$\left(\sum_{j=k}^{\infty} a_j^2\right)^{1/2} = \omega_k.$$

Hence, similarly to the proof of (ii) of Theorem 2, we can conclude from (2) and Kintchin's inequality that $f\in X^p$ and $g\in H_p^{\omega_f}$.

On the other hand

$$\|dS_{2^n}f\|_p \leqslant A_p \left(\sum_{k=1}^{n-1} (2^k a_k)^2 \right)^{1/2} \leqslant A_p \left(\sum_{k=1}^{n-1} (2^k \omega_k)^2 \right)^{1/2},$$

and by (3) we have $E_{2^n}(f, X^p) \leqslant \|f - S_{2^n}f\|_p \leqslant C_p \omega_n \leqslant C_p \alpha_{2^n}$ $(n \in \mathbf{N})$. Consequently, in view of Lemma 2 and Corollary 2 we have $f \in \Sigma_p$. ∎

Proof of Theorem 4. Suppose that $f \in X^p$ $(p = 1$ or $\infty)$ satisfies the hypothesis of (i). Then we have by (10)

$$\begin{aligned} \|dS_{2^n}f\|_p &\leqslant \sum_{k=0}^{n-1} \|d\Delta_k f\|_p \leqslant 4 \sum_{k=0}^{n-1} 2^k \|\Delta_k f\|_p \\ &\leqslant 4 \sum_{k=0}^{n-1} 2^k \omega_p(2^{-k}, f) = O(2^n \alpha_{2^n}) \qquad (n \to \infty). \end{aligned}$$

Since the second assumption of (12) follows obviously from our hypothesis, Corollary 2 implies $\|f - \sigma_k f\|_p = O(\alpha_k)$ as $k \to \infty$.

Next denote $\omega = (\omega_k, k \in \mathbf{N}) \searrow 0$ for which $\sum_{k=0}^{n} \omega_k \neq O(2^n \alpha_{2^n})$ $(n \to \infty)$. Let $(\ell_k, k \in \mathbf{N})$ be as in (15). Set

$$f_k = \begin{cases} a_k \mathscr{I}(\tau_{2^{\ell_{k+1}-1}}(D_{2^{\ell_{k+1}}} - D_{2^{\ell_{k+1}-1}})) & \text{if} \quad p = 1 \\ a_k \mathscr{I}(D_{2^{\ell_{k+1}}} - D_{2^{\ell_{k+1}-1}}) & \text{if} \quad p = \infty, \end{cases}$$

where a_k is determined by $\|f_k\|_p = \omega_{\ell_k}$ $(k \in \mathbf{N})$. Using the same arguments those in the proof of Theorem 2 we obtain

$$f \in H_p^{\omega} \qquad \text{and} \qquad \omega_p(2^n, f) \geqslant \omega_n \quad (n \in \mathbf{N}).$$

By definition we have

$$\operatorname{supp} df_j \cap \operatorname{supp} df_k = \varnothing \qquad (j, k \in \mathbf{N}, j \neq k) \quad \text{if} \quad p = 1$$

and

$$f_k(0) = \|f_k\|_\infty = \omega_{\ell_k} \qquad (k \in \mathbf{N}) \quad \text{if} \quad p = \infty.$$

Therefore, the definition of $(\ell_k, k \in \mathbf{N})$ and (10) imply for $\ell_k \leqslant n < \ell_{k+1}$ $(n, k \in \mathbf{N})$ that

$$\begin{aligned} \|dS_{2^{\ell_k}}f\|_p &= \sum_{j=0}^{k-1} \|df_j\|_p \geqslant C \sum_{j=0}^{k-1} 2^{\ell_j+1} \|f_j\|_p \\ &= C \sum_{j=0}^{k-1} 2^{\ell_j+1} \omega_{\ell_j} \qquad (n \in \mathbf{P}). \end{aligned}$$

Using the same arguments as those in the proof of (ii) of Theorem 3 we obtain $f \notin \Sigma_p$.

In order to show (iii) set $f \in X^p$ with $\|f - \sigma_k f\|_p = O(\alpha_n)$ as $k \to \infty$. Then Corollary 2 and (10) imply

$$2^k \|\Delta_k f\|_p \leqslant C \|d\Delta_k f\|_p \leqslant C \|dS_{2^n} f\|_p \leqslant C 2^n \alpha_{2^n} \qquad (0 < k < n).$$

In particular, (2) implies

$$2^k \omega_p(2^{-k}, f) \leqslant 2 \cdot 2^k \left(\sum_{j=k}^{n-1} \|\Delta_j f\|_p + \|f - S_{2^n} f\|_p \right)$$
$$\leqslant C 2^n \alpha_{2^n} \qquad (k, n \in \mathbf{N}, k < n),$$

which was to be proved.

Finally, let $g \in \Omega_p^{**}$ and $\omega_n = \omega_p(2^{-n}, f)$ $(n \in \mathbf{N})$. To prove (iv) it suffices to show the existence of an $f \in \Sigma_p$ for which

$$\omega_p(2^{-n}, f) \geqslant C\omega_n \qquad (n \in \mathbf{N}).$$

To this end first let $p = \infty$, and denote by $(f_k, k \in \mathbf{N})$, $(\ell_k, k \in \mathbf{N})$ the same sequences as those in (15) and (16). Thus $g \in H_\infty^{\omega_f}$. If $\ell_k \leqslant n < \ell_{k+1}$ $(k, n \in \mathbf{N})$ then by definition we have

$$\|dS_{2^n} f\|_\infty = \max_{0 < j < k} \|df_j\|_\infty \leqslant \max_{0 < j < k} C 2^{\ell_j} \omega_{\ell_j} = O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

If $p = 1$ then let $(\ell_k, k \in \mathbf{N})$ be as before. Set $f_0 = \omega_1 r_0$, and define f_k inductively as

$$df_k = a_k r_{\ell_{k+1}-1} \sum_{j=0}^{k-1} df_j,$$

where a_k is determined by $\|f_k\|_1 = \omega_{\ell_k} (k \in \mathbf{N})$. Clearly $f = \sum_{k=0}^{\infty} f_k \in X^1$ and $g \in H_1^{\omega_f}$. On the other hand, it can be proved by induction that if $\ell_k \leqslant n < \ell_{k+1}$ $(k, n \in \mathbf{N})$ then

$$\|dS_{2^n} f\|_1 = \max_{0 < j < k} \|df_j\|_1 \leqslant \max_{0 < j < k} C 2^{\ell_j} \omega_{\ell_j} = O(2^n \alpha_{2^n}) \qquad (n \to \infty).$$

Indeed, we have from the construction of f_k that

$$\|dS_{\ell_{k+1}} f\|_1 = \begin{cases} \|df_{k+1}\|_1 & \text{if} \quad a_{k+1} > 1 \\ \|dS_{\ell_k} f\|_1 & \text{if} \quad a_{k+1} \leqslant 1. \end{cases}$$

This completes the proof of Theorem 4. ∎

REMARKS

In this section we show some of the consequences of our results. The (i) parts of Theorems 2–4 can be used to estimate the rate of convergence of the Cesàro means of an arbitrary $f \in X^p$ $(1 \leqslant p \leqslant \infty)$. However, that rate may not be the best possible for that individual function, but it is for the Hölder class generated by f. This follows from the (ii) parts of the theorems. In particular, if $\omega_p(2^{-n}, f) = 2^{-n\gamma}$ $(0 < \gamma \leqslant 1)$—for the existence of such a function see [3, 8]—then $H_p^{\omega_f} = \mathrm{Lip}(\gamma, X^p)$. It is easy to see that as a consequence of Theorems 2–4 we receive the well-known result [12, 13], i.e., $\|f - \sigma_n f\|_p = O(n^{-\gamma})$ $(0 < \gamma < 1)$ if and only if $f \in \mathrm{Lip}(\gamma, X^p)$. It is known [12, 13], that $f \in \mathrm{Lip}(1, X^p)$ implies $\|f - \sigma_n f\|^p = O(\log n/n)$ $(n \to \infty)$. Applying our results for this situation we can conclude that this is the best possible estimation if $p = 1$, ∞, but not for the other cases. Namely the following theorem is true.

THEOREM 5. *Let* $f \in \mathrm{Lip}(1, X^p)$ $(1 \leqslant p \leqslant \infty)$.

(i) *If* $p = 1$ *or* ∞ *then*

$$\|f - \sigma_n f\|_p = O\left(\frac{\log n}{n}\right) \qquad (n \to \infty).$$

(ii) *If* $1 < p \leqslant 2$ *then*

$$\|f - \sigma_n f\|_p = O\left(\frac{(\log n)^{1/p}}{n}\right) \qquad (n \to \infty).$$

(iii) *If* $2 \leqslant p < \infty$ *then*

$$\|f - \sigma_n f\|_p = O\left(\frac{(\log n)^{1/2}}{n}\right) \qquad (n \to \infty).$$

These results cannot be improved.

The problem of characterizing the Favard (saturation) class of the Cesàro summation via the X^p modulus of continuity was posed by Móricz and Siddiqi in [6]. In our case the saturation class is the collection of functions for which $\|f - \sigma_n f\|_p = O(n^{-1})$ $(n \to \infty)$. The solution of the above problem reads as follows.

THEOREM 6. *Let* $f \in X^p$ $(1 \leqslant p \leqslant \infty)$.

(i) *If*

$$\sum_{k=0}^{\infty} k^{q-1} \omega_p^q(n^{-1}, f) < \infty,$$

where $q = p$ *if* $1 < p \leqslant 2$, $q = 2$ *if* $2 \leqslant p < \infty$, *and* $q = 1$ *if* $p = 1$ *or* ∞, *then*

$$\|f - \sigma_n f\|_p = O(n^{-1}) \qquad (n \to \infty).$$

(ii) *The above conditions cannot be weakened.*

(iii) *If* $\|f - \sigma_n f\|_p = O(n^{-1})$ $(n \to \infty)$, *then*

$$\sum_{k=0}^{\infty} k^{q-1} \omega_p^q(n^{-1}, f) < \infty,$$

where $q = 2$ *if* $1 < p \leqslant 2$, $q = p$ *if* $2 \leqslant p < \infty$, *and*

$$f \in \mathrm{Lip}(1, X^p)$$

if $p = 1$ *or* ∞.

(iv) *The result of* (iii) *cannot be improved.*

We note that the case (i) follows from the results of Móricz and Siddiqi [6] for $p = 1$ or ∞.

Our final remark concerns the relation between the type of conditions in the above theorems for the cases $1 < p < \infty$ and $p = 1$ or ∞. It can be seen that the L^1 and C_W spaces do not act as limit cases of L^p $(1 < p < \infty)$. This is especially clear in Theorems 5 and 6. To fill this gap one must take the dyadic Hardy and VMO spaces. It is not hard to check that our technique applied for $1 < p < \infty$ can be extended for these spaces. The corresponding results for the dyadic Hardy space can be obtained formally from the L^p $(1 < p < \infty)$ results by letting $p \to 1$. Similarly, the results for the VMO space are obtained by letting $p \to \infty$. The dyadic Hardy and VMO spaces behave similarly, for instance, in problems connected with embedding relations [4].

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